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Membrane transport of electrolyte ions and time-dependent membrane potential

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Equations are derived for the transport of a symmetrical electrolyte, consisting of cations and anions of equal valency, through a neutral membrane that separates two solutions of finite volume under quasi-steady-state conditions. The time-dependent membrane potential produced by the flow of ions is taken into account. Deviation of the time course of the solute concentrations from that of neutral solutes is found to be determined by the permeability ratio of cations and anions (when this ratio equals unity, the derived membrane transport equations reduce to those for neutral substances). Simple approximate expressions for the solute concentrations and of the membrane potential as functions of time are proposed, which are in excellent agreement with the exact numerical results.

1. Introduction

Ion transport is one of the fundamental characteristics of membranes [1]. When a membrane separates two solutions, 1 and 2, both being of finite volume, and is permeable to solutes, the time course of the solute concentrations in the respective solutions under steady-state conditions [2] is often described by (see, e.g., ref. 3):

$$V_1 \frac{dC_1}{dt} = -PA(C_1 - C_2), \tag{1}$$

$$V_2 \frac{dC_2}{dt} = -PA(C_2 - C_1), \tag{2}$$

where A is the membrane area, P the permeability of the solutes, $C(t)$ the solute concentration at time t and V the solution volume, subscripts 1 and 2 referring to solutions 1 and 2, respectively. Under the initial conditions

$$C_1(0) = C_0, \tag{3}$$

$$C_2(0) = 0, \tag{4}$$

the solution to eqs. 1 and 2 gives

$$C_1(t) = C_0(\phi e^{-\alpha t} + 1 - \phi), \tag{5}$$

$$C_2(t) = C_0(1 - \phi)(1 - e^{-\alpha t}), \tag{6}$$

where

$$\phi = V_2/(V_1 + V_2) \tag{7}$$

is the volume fraction of solution 2 and α is defined by

$$\alpha = PA(1/V_1 + 1/V_2). \tag{8}$$

Eqs. 1 and 2, however, can be applied only to the case where solute molecules are electrically neutral. For dealing with charged solutes, one must take into account: (i) the flow of counterions, since the net electric current must be zero

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within the membrane in the absence of external electric fields; and (ii) the membrane potential established between solutions 1 and 2, which accelerates the slower ions and retards the faster ions. This membrane potential, which depends on time, is of particular interest, since the membrane potential has usually been considered under the conditions that the solute concentration in solutions 1 and 2 remain constant with respect to time and therefore that the membrane potential is independent of time except during the initial transient time period [4–7].

In the present paper, taking into account both factors described above, we generalize the expressions for membrane transport (eqs. 1 and 2) to cover the case of charged species in the absence of external electric fields under quasi-steady-state conditions.

2. Theory

Consider a quasi-steady flow of symmetrical electrolyte ions of valency z across a planar neutral membrane of thickness d and area A separating two solutions 1 and 2, the volumes of which are V_1 and V_2, respectively. We take the x-axis perpendicular to the membrane, its surfaces being placed at $x = 0$ and $x = d$ (fig. 1). The flows of cations and anions, which we denote by J_+ and J_-, respectively, are given by

$$J_\pm(x) = -D_\pm\left(\frac{\partial C_\pm(x)}{\partial x} \pm \frac{C_\pm(x)ze}{kT}\frac{\partial \psi}{\partial x}\right), \quad (9)$$

where $D_\pm$ are the respective diffusion coefficients of cations and anions, $C_\pm(x)$ the corresponding concentrations of cations and anions within the membrane, e the elementary electric charge, k Boltzmann's constant, T the absolute temperature and $\psi(x)$ the electric potential at position x relative to the bulk phase of solution 1 (where ψ is put equal to zero). Under a quasi-steady condition, $\operatorname{div} J_\pm(x) = \mathrm{d}J_\pm/\mathrm{d}x = 0$ and the net electric current must be zero in the absence of an external field so that $J_+ = J_- =$ constant. We term this constant J, viz.,

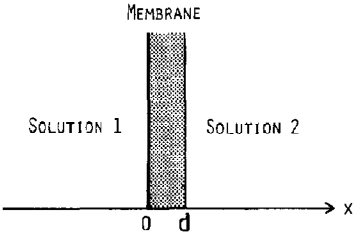

Fig. 1. A membrane of area A and thickness d separating two solutions 1 and 2, of volumes V_1 and V_2, respectively.

$$J_+ = J_- = J. \quad (10)$$

Following Goldman [4] and Hodgkin and Katz [5], we assume that the electric field is constant within the membrane. Then, eq. 9 can be integrated to give

$$J_\pm = \mp\frac{zeE_\mathrm{m}}{kT}P_\pm\frac{C_{\pm2}\exp(\pm zeE_\mathrm{m}/kT) - C_{\pm1}}{\exp(\pm zeE_\mathrm{m}/kT) - 1}, \quad (11)$$

Here, $C_{\pm1}$ and $C_{\pm2}$ are the respective ion concentrations in solutions 1 and 2 and are related to the corresponding concentrations within the membrane by

$$C_\pm(+0) = b_\pm C_{\pm1}, \quad (12)$$

$$C_\pm(\mathrm{d} - 0) = b_\pm C_{\pm2}, \quad (13)$$

with $b_\pm$ denoting the partition coefficients of cations and anions, respectively, between the membrane and solution phases, E_m representing the membrane potential, i.e., the potential of solution 2 relative to that of solution 1; and P_+ and P_- correspond to the membrane permeabilities of cations and anions, defined by

$$P_\pm = b_\pm D_\pm/d. \quad (14)$$

As required by electroneutrality, the concentrations of cations and anions are equal in both solutions 1 and 2. We designate as C_1 and C_2 the electrolyte concentrations in solutions 1 and 2, respectively, viz.,

$$C_{+1} = C_{-1} = C_1, \quad (15)$$

$$C_{+2} = C_{-2} = C_2. \quad (16)$$

By substituting eq. 11 into eq. 10 and using eqs. 15 and 16, we find that the membrane potential E_m is given by

$$E_m = \frac{kT}{ze}\ln\left(\frac{P_+C_1 + P_-C_2}{P_+C_2 + P_-C_1}\right). \tag{17}$$

The time course of the concentrations C_1 and C_2 under the initial conditions (eqs. 3 and 4) can be described by Fick's first law, viz.,

$$V_1\,\mathrm{d}C_1(t)/\mathrm{d}t = -V_2\,\mathrm{d}C_2(t)/\mathrm{d}t = JA, \tag{18}$$

where the amount of solutes within the membrane has been neglected. Combining eqs. 11, 17 and 18, we obtain

$$\mathrm{d}C_1(t)/\mathrm{d}t = -\alpha\frac{zeE_m}{kT}\frac{2r}{r^2-1}\left[(1-\phi)C_0 + (2\phi-1)C_1(t)\right], \tag{19}$$

$$C_2(t) = [(1-\phi)/\phi][C_0 - C_1(t)], \tag{20}$$

with

$$\alpha = (P_+ + P_-)/2\cdot A(1/V_1 + 1/V_2), \tag{21}$$

$$r = P_+/P_-, \tag{22}$$

where ϕ is defined by eq. 8. Eqs. 17, 19 and 20 completely determine the time course of the electrolyte concentrations and of the membrane potential.

3. Results and discussion

We have derived a system of equations describing the time course of electrolyte concentrations on both sides of a membrane (eqs. 19 and 20) and the time-dependent membrane potential (eq. 17). As a result of electroneutrality (eqs. 15 and 16), the concentrations of cations and anions are always equal and their time course is given by $C_1(t)$ and $C_2(t)$ in solutions 1 and 2, respectively. The first-order differential equation (eq. 19) can be easily integrated by usual numerical methods. Some results from the calculation are displayed in figs. 2–4, which depict the electrolyte concentrations in solutions 1 and 2, $C_1(t)$ and $C_2(t)$, and the membrane potential $E_m(t)$ as functions of the

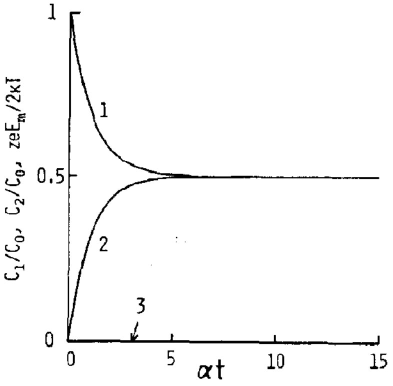

Fig. 2. Electrolyte concentrations in solution 1, $C_1(t)$, and solution 2, $C_2(t)$, and membrane potential, E_m, as functions of scaled time αt. Curves: 1, $C_1(t)/C_0$; 2, $C_2(t)/C_0$; 3, $zeE_m/2kT$. Calculated with volume fraction $\phi = 0.5$ and $P_+/P_- = 1$.

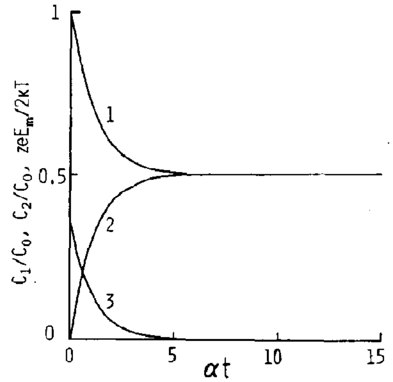

Fig. 3. Same as fig. 2, except $P_+/P_- = 2$. The approximate results (eqs. 27–29) agree with the exact results within the linewidth.

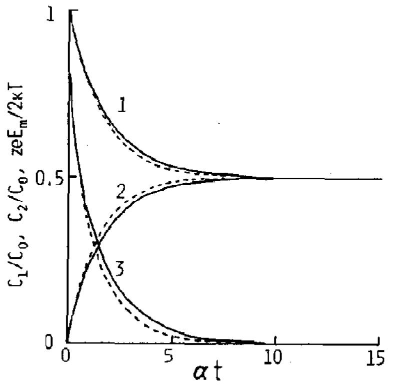

Fig. 4. Same as fig. 2, except $P_+/P_- = 5$. Dashed lines are the approximate results (eqs. 27–29).

scaled time αt. In the case of $P_+ = P_-$ (fig. 2), a membrane potential is not evoked, so that the result in this case agrees with that of membrane transport of neutral substances (eqs. 5 and 6).

When $P_+ \neq P_-$, as shown in figs. 3 and 4, a membrane potential is produced between solutions 1 and 2, which accelerates slower ions (or ions of lower permeability) and retards faster species (or those of higher permeability), reducing the rate of change in $C_1(t)$ and $C_2(t)$. The magnitude of the membrane potential E_m reaches its maximum at $t = 0$, which is

$$E_m(0) = (kT/ze) \ln r, \tag{23}$$

and tends to zero as $t \to \infty$. As the difference between P_+ and P_- increases, the magnitude of the membrane potential becomes greater, leading to slower changes in $C_1(t)$ and $C_2(t)$. Note that $C_1(t)$, $C_2(t)$ and $E_m(t)$ depend on the relative ratio of P_+ and P_-. Therefore, the results for $P_+/P_- = 2$ (fig. 3) and 5 (fig. 4) are equivalent to those for $P_+/P_- = 1/2$ and $1/5$, respectively, except that E_m reverses its sign. Note also that, since $E_m \to 0$ as $t \to \infty$, $C_1(t)$ and $C_2(t)$ tend to the same limiting values as in the case of neutral substances (see eqs. 5 and 6), viz.,

$$\lim_{t\to\infty} C_1(t) = \lim_{t\to\infty} C_2(t) = C_0(1-\phi). \tag{24}$$

The initial slope of d log $C_1/\mathrm{d}t$, which can be easily related with experimental data, is found to be

$$|\mathrm{d} \log C_1(t)/\mathrm{d}t\,|_{t=0} = \alpha f \phi/2.303, \tag{25}$$

with

$$f = \left(2r/(r^2-1)\right) \ln r. \tag{26}$$

Since $f = 1$ for neutral solutes ($z = 0$), the factor f may be interpreted as the ratio of the initial slope of d log $C_1(t)/\mathrm{d}t$ for charged solutes to that for neutral solutes. Fig. 5 shows f as a function of r, which is symmetrical with respect to $r = 1$. At $r = 1$, f equals unity, which means that the case where $P_+ = P_-$ is equivalent to that for neutral solutes. As the value of f departs from unity, f decreases. At $r = 10$, for example, $f = 0.47$, i.e., the initial slope of d log $C_1/(t)/\mathrm{d}t$ is decreased to 47% of that in the case of neutral substances.

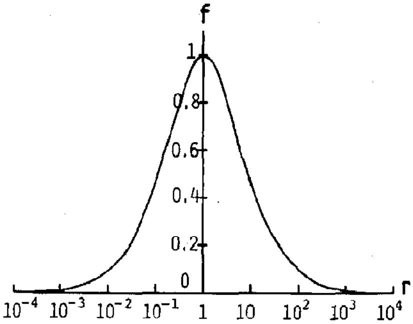

Fig. 5. The ratio, f, of the initial slope, $|\mathrm{d} \log C_1(t)/\mathrm{d}t|_{t=0}$, to that for neutral substances as a function of $r(=P_+/P_-)$.

Eq. 17 is of the same form as the Goldman-Hodgkin-Kats (GHK) equation for the membrane potential [4,5]. However, it must be emphasized that the GHK formula refers to the membrane potential under steady-state conditions, in which C_1 and C_2 are constant and independent of time t, whereas in our treatment C_1 and C_2 and thus E_m all vary with respect to time.

Figs. 3 and 4 suggest that the time course of $C_1(t)$ and $C_2(t)$ may be approximated by an exponential-type function, similar to eqs. 5 and 6. If the time constant α in eqs. 5 and 6 is replaced by αf, which is suggested by eq. 25, we obtain the following approximate expressions:

$$C_1(t) = C_0(\phi e^{-\alpha f t} + 1 - \phi), \tag{27}$$

$$C_2(t) = C_0(1-\phi)(1 - e^{-\alpha f t}). \tag{28}$$

Substituting eqs. 27 and 28 into eq. 17, we find

$$\begin{aligned} E_m(t) = \frac{kT}{ze} \ln\Big[&\{(1-\phi)(1+r) \\ &+ \{r\phi - (1-\phi)\} \exp(-\alpha f t)\} \\ &\times \{(1-\phi)(1+r) + \{\phi - r(1-\phi)\} \\ &\times \exp(-\alpha f t)\}^{-1}\Big]. \end{aligned} \tag{29}$$

The approximate results (eqs. 27–30), represented by the dashed lines in fig. 4, are found to be in excellent agreement with the exact results (solid lines), unless the ratio P_+/P_- is not very different from unity. When $P_+/P_- = 2$ (fig. 3), in particular, the approximate and exact results agree within the linewidth.

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